Devoir 4

Partie 1

1) $z = e^{i\frac{\pi}{7}}$

$$z^6 - z^5 + z^4 - z^3 + z^2 - z + 1 = \sum_{k=0}^{6} (-1)^k z^k = \sum_{k=0}^{6} (-z)^k = \frac{1-(-z)^7}{1-(-z)} \quad (-z \neq 1)$$

$$= \frac{1-(-e^{i\frac{\pi}{7}})^7}{1+e^{i\frac{\pi}{7}}} \qquad -e^{i\frac{\pi}{7}} \neq 1$$

$(-1)^7 = -1$ et Moivre $(e^{i\frac{\pi}{7}})^7 = e^{i\pi} = (-1)$

$$= \frac{1 + e^{i\pi}}{1+e^{i\frac{\pi}{7}}}$$

$$= 0$$

) $(1-z^3)(z^3-z^2+z) = z^3 - z^2 + z - z^6 + z^5 - z^4$

or d'après le 1), $z^6 - z^5 + z^4 - z^3 + z^2 - z + 1 = 0$

donc $-z^6 + z^5 - z^4 + z^3 - z^2 + z = 1$

d'où $(1-z^3)(z^3-z^2+z) = 1$

ainsi $z^3 - z^2 + z = \dfrac{1}{1-z^3}$

) Soit $u \in \mathbb{U}$, on suppose $u \neq 1$

on peut poser $u = e^{i\theta}$ avec $\theta \in \mathbb{R}$ et $\theta \neq 2k\pi, \forall k \in \mathbb{Z}$

$$\frac{1}{1-u} = \frac{1}{1-e^{i\theta}} = \frac{1}{e^{i\frac{\theta}{2}}(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}})} = \frac{e^{-i\frac{\theta}{2}}}{-2i\sin\frac{\theta}{2}} = -\frac{1}{2i} e^{-i\frac{\theta}{2}} \times \frac{1}{\sin\frac{\theta}{2}}$$

or $\dfrac{1}{2i} = \dfrac{1}{2} \times \dfrac{1}{i} = \dfrac{1}{2}(-i)$

donc
$$\frac{1}{1-u} = \frac{i}{2} e^{-i\frac{\theta}{2}} \frac{1}{\sin\frac{\theta}{2}} = \frac{1}{2\sin\frac{\theta}{2}}\left(i\left(\cos\left(-\frac{\theta}{2}\right) + i\sin\left(-\frac{\theta}{2}\right)\right)\right)$$

$$= \frac{1}{2\sin\frac{\theta}{2}}\left(-\sin\left(-\frac{\theta}{2}\right) + i\cos\left(-\frac{\theta}{2}\right)\right)$$

sin impaire, cos paire

$$= \frac{1}{2\sin\theta/2}\left(\sin\left(\frac{\theta}{2}\right) + i\cos\left(\frac{\theta}{2}\right)\right)$$

donc $\mathrm{Re}\left(\dfrac{1}{1-u}\right) = \boxed{\dfrac{\sin(\theta/2)}{2\sin(\theta/2)} = \dfrac{1}{2}}$

4) Ainsi, en prenant $u = z^3 = e^{i\frac{3\pi}{7}}$

$u \in \mathbb{U}$ et $u \neq 1$

donc avec le 3) $\mathrm{Re}\left(\dfrac{1}{1-z^3}\right) = \dfrac{1}{2}$

Mais avec le 2), $\dfrac{1}{1-z^3} = z^3 - z^2 + z = e^{i\frac{3\pi}{7}} - e^{i\frac{2\pi}{7}} + e^{i\frac{\pi}{7}}$

donc $\boxed{\dfrac{1}{2} = \mathrm{Re}\left(\dfrac{1}{1-z^3}\right) = \cos\left(\dfrac{3\pi}{7}\right) - \cos\left(\dfrac{2\pi}{7}\right) + \cos\left(\dfrac{\pi}{7}\right)}$

Partie 2

1) $\mathbb{U}_5 = \{ e^{i\frac{2k\pi}{5}}, k \in [\![0,4]\!] \}$

2) (E) : $P(z) = 0 \iff \dfrac{1}{2i}\left((z+i)^5 - (z-i)^5\right) = 0$

$\iff (z+i)^5 = (z-i)^5$

or $z = i$ n'est pas solution de (E) car $(i+i)^5 \neq (i-i)^5 = 0$

donc $(E) \iff \left(\dfrac{z+i}{z-i}\right)^5 = 1$

$\iff \exists k \in [\![0,4]\!], \quad \dfrac{z+i}{z-i} = e^{i\frac{2k\pi}{5}} = \omega^k$

$\iff \exists k \in [\![0,4]\!] \quad z + i = \omega^k (z - i)$

$\iff \exists k \in [\![0,4]\!] \quad z(1-\omega^k) = -i\omega^k - i$

$\iff \exists k \in [\![1,4]\!] \quad z = \dfrac{-i\omega^k - i}{1-\omega^k}$

pour $k=0$ $\omega^k = 1$ on a $0 = -i-1$ c'est impossible

Soit $k \in [\![1,4]\!]$,

$$\frac{-i\omega^k - i}{1-\omega^k} = \frac{i + i\omega^k}{\omega^k - 1} = i\frac{1 + e^{i\frac{2k\pi}{5}}}{e^{i\frac{2k\pi}{5}} - 1} = i\frac{e^{i\frac{k\pi}{5}}(e^{-i\frac{k\pi}{5}} + e^{i\frac{k\pi}{5}})}{e^{i\frac{k\pi}{5}}(e^{i\frac{k\pi}{5}} - e^{-i\frac{k\pi}{5}})}$$

$$= i\frac{2\cos(\frac{k\pi}{5})}{2i\sin(\frac{k\pi}{5})} = \boxed{\frac{1}{\tan\frac{k\pi}{5}}}$$

3) $P(z) = \dfrac{1}{2i}\left((z+i)^5 - (z-i)^5\right)$

Binôme
$$= \frac{1}{2i}\left(z^5 + 5z^4 i + 10z^3 i^2 + 10z^2 i^3 + 5z i^4 + i^5 - (z^5 - 5z^4 i + 10z^3 i^2 - 10z^2 i^3 + 5z i^4 - i^5)\right)$$

$$= \frac{1}{2i}\left(10z^4 i + 20z^2 i^3 + 2i^5\right)$$

$$= \frac{1}{2i}\left(2i(5z^4 + 10z^2 i^2 + i^4)\right)$$

$$= 5z^4 - 10z^2 + (i^2)^2 \quad (-1)^2 = 1$$

$$= 5z^4 - 10z^2 + 1$$

) On pose $Z = z^2$

on résout $5Z^2 - 10Z + 1 = 0$

$\Delta = 100 - 20 = 80 = 4 \times 20 = 16 \times 5$

$Z_1 = \dfrac{10 + 4\sqrt{5}}{10}$ $\qquad Z_2 = \dfrac{10 - 4\sqrt{5}}{10}$

donc $Z_1 = \dfrac{5 + 2\sqrt{5}}{5}$ $\qquad Z_2 = \dfrac{5 - 2\sqrt{5}}{5}$

Puis $P(z) = 0 \iff z^2 = Z_1$ ou $z^2 = Z_2$

$Z_1 > 0$ et $Z_2 > 0$

$\iff z = \sqrt{\dfrac{5+2\sqrt{5}}{5}}$ ou $z = -\sqrt{\dfrac{5+2\sqrt{5}}{5}}$ ou $z = \sqrt{\dfrac{5-2\sqrt{5}}{5}}$ ou $-\sqrt{\dfrac{5-2\sqrt{5}}{5}}$

Or on a vu au 2) que les solutions sont

$\dfrac{1}{\tan\frac{\pi}{5}}$, $\dfrac{1}{\tan\frac{2\pi}{5}}$, $\dfrac{1}{\tan\frac{3\pi}{5}}$, $\dfrac{1}{\tan\frac{4\pi}{5}}$

or $\tan\left(\dfrac{3\pi}{5}\right) = \tan\left(\pi - \dfrac{2\pi}{5}\right) \overset{\tan\ \pi\text{-périodique}}{=} \tan\left(-\dfrac{2\pi}{5}\right) \overset{\tan\ \text{impaire}}{=} -\tan\left(\dfrac{2\pi}{5}\right)$

de même $\tan\left(\dfrac{4\pi}{5}\right) = \tan\left(\pi - \dfrac{\pi}{5}\right) = -\tan\left(\dfrac{\pi}{5}\right)$

donc les solutions sont : $\dfrac{1}{\tan\pi/5}$, $\dfrac{1}{\tan 2\pi/5}$, $-\dfrac{1}{\tan\pi/5}$, $-\dfrac{1}{\tan\frac{2\pi}{5}}$

or $0 < \dfrac{\pi}{5} < \dfrac{2\pi}{5} < \dfrac{\pi}{2}$ et tan est strictement croissante sur $]0, \frac{\pi}{2}[$

donc $0 < \tan\dfrac{\pi}{5} < \tan\dfrac{2\pi}{5}$

d'où $0 < \dfrac{1}{\tan\frac{2\pi}{5}} < \dfrac{1}{\tan\frac{\pi}{5}}$; puis $0 < 5 - 2\sqrt{5} < 5 + 2\sqrt{5}$ donc $\sqrt{\dfrac{5-2\sqrt{5}}{5}} < \sqrt{\dfrac{5+2\sqrt{5}}{5}}$

ainsi $\dfrac{1}{\tan\frac{\pi}{5}} = \sqrt{\dfrac{5+2\sqrt{5}}{5}}$ et $\dfrac{1}{\tan\frac{2\pi}{5}} = \sqrt{\dfrac{5-2\sqrt{5}}{5}}$

d'où $\tan\dfrac{\pi}{5} = \sqrt{\dfrac{5}{5+2\sqrt{5}}} = \sqrt{\dfrac{5(5-2\sqrt{5})}{25-20}}$ et $\tan\dfrac{2\pi}{5} = \sqrt{\dfrac{5}{5-2\sqrt{5}}} = \sqrt{\dfrac{5(5+2\sqrt{5})}{25-20}}$

$\boxed{\tan\frac{\pi}{5} = \sqrt{5-2\sqrt{5}}}$ $\qquad \boxed{\tan\frac{2\pi}{5} = \sqrt{5+2\sqrt{5}}}$

6) $\forall t \in]0, \frac{\pi}{2}[$ $\quad 1 + \tan^2 t = \dfrac{1}{\cos^2 t} = \tan' t$

donc $\cos^2 t = \dfrac{1}{1+\tan^2 t}$ et ici $\cos^2\dfrac{\pi}{5} = \dfrac{1}{1+5-2\sqrt{5}} = \dfrac{1}{6-2\sqrt{5}}$

mais $\cos\dfrac{\pi}{5} > 0$ donc $\boxed{\cos\frac{\pi}{5} = \frac{1}{\sqrt{6-2\sqrt{5}}} = \sqrt{\frac{6+2\sqrt{5}}{36-20}} = \sqrt{\frac{3+\sqrt{5}}{8}}}$

Exercice 2

1) a) Soient p, n deux entiers naturels tels que $0 \le p \le n-1$

Alors $\binom{n}{p} + \binom{n}{p+1} = \binom{n+1}{p+1}$

b) Soit $n \in \mathbb{N}$, $n \ge 2$, $\forall k \in [\![3, n]\!]$, (d'après le 1) a)), $\binom{k}{2} + \binom{k}{3} = \binom{k+1}{3}$

Ainsi $\sum_{k=2}^{n} \binom{k}{2} = \binom{2}{2} + \sum_{k=3}^{n} \binom{k}{2} = 1 + \sum_{k=3}^{n} \left(\binom{k+1}{3} - \binom{k}{3} \right)$

$\underset{\text{(telescopage)}}{=} 1 + \binom{n+1}{3} - \binom{3}{3} = \binom{n+1}{3}$

c) Ainsi, $\sum_{k=2}^{n} \left(2\binom{k}{2} + k \right) \underset{\text{linéarité}}{=} 2 \sum_{k=2}^{n} \binom{k}{2} + \sum_{k=2}^{n} k$

$= 2\binom{n+1}{3} + \left(\frac{n(n+1)}{2} - 1 \right)$

Mais par ailleurs, $2\binom{k}{2} + k = 2 \frac{k!}{(k-2)!\,2!} + k = k(k-1) + k$

$= k^2 - k + k = k^2$

d'où $\forall k \in [\![2, n]\!]$, $2\binom{k}{2} + k = k^2$

et $\sum_{k=2}^{n} k^2 = 2\binom{n+1}{3} + \frac{n(n+1)}{2} - 1$

donc $\sum_{k=1}^{n} k^2 = 1 + \sum_{k=2}^{n} k^2 = 2\binom{n+1}{3} + \frac{n(n+1)}{2}$

$= 2 \frac{(n+1)n(n-1)}{3!} + \frac{3n(n+1)}{6}$

$= \frac{2(n+1)n(n-1) + 3n(n+1)}{6} = \frac{n(n+1)[2n-2+3]}{6}$

$$\boxed{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}$$

e) Soit $n \in \mathbb{N}^*$

$\sum_{1 \le i \le j \le n} i = \sum_{i=1}^{n} \sum_{j=i}^{n} i = \sum_{i=1}^{n} i \left(\sum_{j=i}^{n} 1 \right) = \sum_{i=1}^{n} i(n-i+1)$

$= \sum_{i=1}^{n} (i(n+1) - i^2) = (n+1) \sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2$

$= (n+1) \frac{n(n+1)}{2} - S_n$ (avec $S_n = \sum_{i=1}^{n} i^2$)

or, $\sum_{1 \le i \le j \le n} i = \sum_{j=1}^{n} \sum_{i=1}^{j} i = \sum_{j=1}^{n} \frac{j(j+1)}{2} = \frac{1}{2} \sum_{j=1}^{n} j^2 + \frac{1}{2} \sum_{j=1}^{n} j$

$= \frac{1}{2} S_n + \frac{1}{2} \frac{n(n+1)}{2}$

Ainsi $\frac{1}{2} S_n + \frac{1}{2} \frac{n(n+1)}{2} = \frac{1}{2} n(n+1)^2 - S_n$

$\frac{3}{2} S_n + \frac{1}{2} \frac{n(n+1)}{2} = \frac{1}{2} n(n+1)^2$

$3 S_n = n(n+1)^2 - \frac{n(n+1)}{2} = n(n+1)\left[n+1-\frac{1}{2}\right]$

donc $$\boxed{S_n = \frac{n(n+1)(n+\frac{1}{2})}{3} = \frac{n(n+1)(2n+1)}{6}}$$

Exercice 3

Soit $\alpha \in \mathbb{R}$ et $n \in \mathbb{N}$, $n \ge 3$

a) $S_1 = \sum_{k=0}^{n-1} e^{i\frac{2k\pi}{n}} = 0$ (somme des racines n-ièmes de 1)

$S_2 = \sum_{k=0}^{n-1} e^{i\frac{4k\pi}{n}} = \sum_{k=0}^{n-1} \left(e^{i\frac{4\pi}{n}} \right)^k \underset{e^{i\frac{4\pi}{n}} \ne 1}{\overset{(SG)}{=}} \frac{1 - \left(e^{i\frac{4\pi}{n}}\right)^n}{1 - e^{i\frac{4\pi}{n}}} = \frac{1 - e^{i4\pi}}{1 - e^{i\frac{4\pi}{n}}} = 0$

pour tout $n \ge 3$

$\left(\frac{4\pi}{n} \in \left]0, \frac{4\pi}{3}\right] \right)$

b) $\sum_{k=0}^{n-1} \cos\left(\frac{2k\pi}{n}\right) = Re(S_1) = Re(0) = 0$

$\sum_{k=0}^{n-1} \sin\left(\frac{2k\pi}{n}\right) = Im(S_1) = Im(0) = 0$

$\sum_{k=0}^{n-1} \cos\left(\frac{4k\pi}{n}\right) = Re(S_2) = 0$

$\sum_{k=0}^{n-1} \sin\left(\frac{4k\pi}{n}\right) = Im(S_2) = 0$

2) a) $i(t) = \sum_{k=0}^{n-1} i_k(t) = \sum_{k=0}^{n-1} I\sqrt{2} \sin\left(\omega t + \frac{2k\pi}{n}\right)$

$\underset{\text{(linéarité)}}{=} I\sqrt{2} \sum_{k=0}^{n-1} \sin\left(\omega t + \frac{2k\pi}{n}\right)$

1 c) $\sum_{k=0}^{n-1} \cos\left(\alpha + \frac{4k\pi}{n}\right) = Re\left(\sum_{k=0}^{n-1} e^{i(\alpha + \frac{4k\pi}{n})} \right) \underset{\text{linéarité}}{=} Re\left(e^{i\alpha} \sum_{k=0}^{n-1} e^{i\frac{4k\pi}{n}} \right)$

$= Re\left(e^{i\alpha} S_2 \right) = Re\left(e^{i\alpha} \times 0 \right) = 0$

$\sum_{k=0}^{n-1} \sin\left(\alpha + \frac{2k\pi}{n}\right) = Im\left(\sum_{k=0}^{n-1} e^{i(\alpha + \frac{2k\pi}{n})} \right)$

$= Im\left(e^{i\alpha} \sum_{k=0}^{n-1} e^{i\frac{2k\pi}{n}} \right)$

$= Im\left(e^{i\alpha} S_1 \right) = 0$

2) a) (suite)

Ainsi avec $\alpha = \omega t$ on obtient $\boxed{i(t) = I\sqrt{2} \times 0 = 0}$

2) b) $P(t) = \sum_{k=0}^{n-1} v_k(t)\, i_k(t) = \sum_{k=0}^{n-1} I\sqrt{2} \sin\left(\omega t + \frac{2k\pi}{n}\right) V\sqrt{2} \sin\left(\omega t + \frac{2k\pi}{n}\right.$

$\underset{\text{linéarité}}{=} 2IV \sum_{k=0}^{n-1} \underbrace{\sin\left(\omega t + \frac{2k\pi}{n}\right)}_{a} \sin\left(\underbrace{\omega t + \frac{2k\pi}{n} - \varphi}_{b}\right)$

or $\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b) \right)$

donc $P(t) = 2IV \times \frac{1}{2} \sum_{k=0}^{n-1} \left(\cos(\varphi) - \cos\left(2\omega t + \frac{4k\pi}{n} - \varphi\right) \right)$

$= IV \left(\sum_{k=0}^{n-1} \cos\varphi - \sum_{k=0}^{n-1} \cos\left(2\omega t - \varphi + \frac{4k\pi}{n}\right) \right)$ avec $\alpha = 2\omega t - \varphi$ dans le 1) c)

$= IV \left(n \cos\varphi - 0 \right)$

$$\boxed{P(t) = n\, IV \cos\varphi}$$