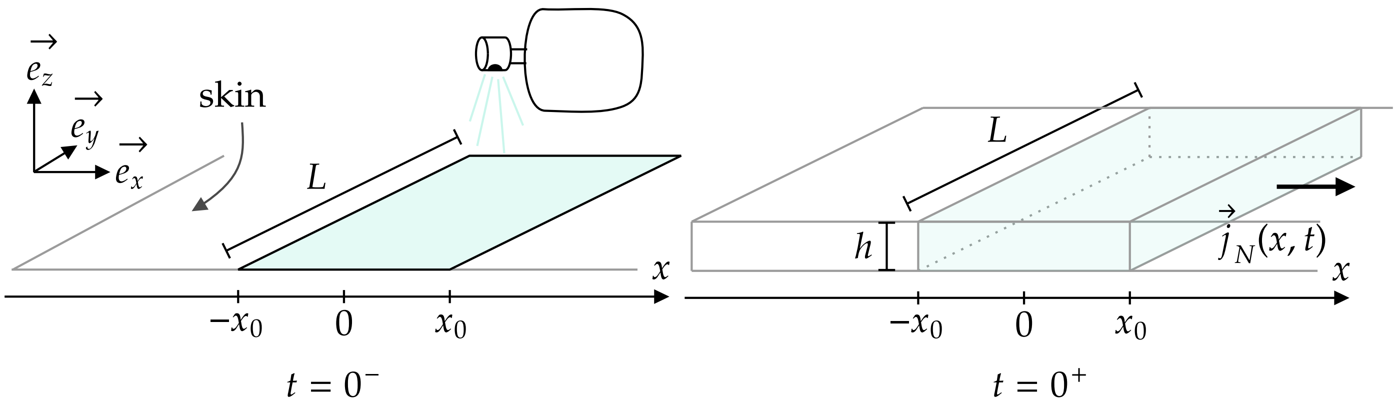
Duration : 2 h 30. The use of any calculating device is forbidden. Any affirmation must be justified.

I - Diffusion of a perfume

(33% of the points)

After initially applying perfume locally on the skin on a surface $2x_0 \times L$, we model here 1D diffusion of perfume molecules in air. Let $n(x,t)$ be the number of perfume molecules per volume of air. Due to slow evaporation the deposit of perfume liberates perfume molecules within the air on top of the $2x_0 \times L$ surface. We note α the constant number of perfume molecules added per unit of volume and per unit of time within the air layer of $-x_0 \leq x \leq x_0$.

We note D the diffusion coefficient of perfume in air, and $\overrightarrow{j}_n(x,t)$ the diffusion flux density. We assume that the plane (O, $\overrightarrow{e_y}$,$\overrightarrow{e_z}$) is Π^+, that is plane of symmetry, for n and for $\overrightarrow{j}_n$.



Q1. Define Fick's law, then justify that $\overrightarrow{j_n}(x,t) = j_n(x,t)\overrightarrow{e_x}$ with $j_n(x,t) = \overrightarrow{j_n}(x,t).\overrightarrow{e_x}$.

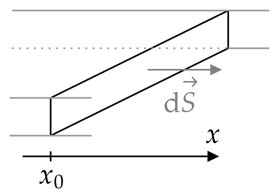
Q2. Establish rigorously the material balance for a layer of air between x and $x + \mathrm{d}x$ between t and $t + \mathrm{d}t$, first within $x \in [-x_0, x_0]$, then (without details) for $x \in \mathbb{R}\backslash[-x_0, x_0]$.

We now study the diffusion in steady-state. This model will be simplistic but will allow some conclusions.

Q3. Establish the expression of $n(x)$ for $x < -x_0$,$-x_0 \leq x \leq x_0$ and $x > x_0$, with 6 unknowns that we leave undetermined for now.

Q4. Demonstrate that if $x \in [-x_0, x_0]$, $n(x) = n_0 - \frac{\alpha}{2D}x^2$.

Q5. Express the number $\dot{N}$ of perfume particles that pass from left to right through the $L \times h$ surface at $x = +x_0$ as a function of α, x_0, L and h.



A typical bottle of perfume contains 50 mL of liquid perfume of molar mass $M \simeq 100$ g.mol^{-1} and volumetric mass $\mu \simeq 10^3$ kg.m^{-3}. Such a bottle lasts about 6 months, for 2 sprays a day, each spray lasting around 5 hours before completely evaporating. One can estimate $h = 5$ mm, $L = 2$ cm and $x_0 = 1$ cm.

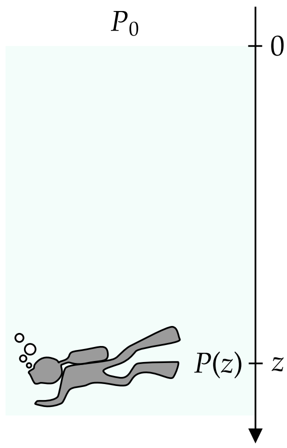
Q6. Establish an order of magnitude for α using the given data.

The diffusion coefficient of perfume within air is $D \simeq 3 \times 10^{-5}$ m^2.s^{-1}.

Q7. Estimate the time Δt for perfume molecules to diffuse over 1 meter of air. Relate your result to everyday life observations, and to other physical phenomenons.

II - Scuba diving accident : growth of gas bubbles

(67% of the points)



The 3 subparts of this problem are independent.

During scuba diving, the body is exposed to increasing hydrostatic pressure as depth increases. At higher pressures, more gases dissolve in living tissue. If the diver rises to the surface very quickly, the gas trapped in the tissues has no time to return to the blood and lungs, and turns into gas bubbles that can become lethal.
Throughout this study, we suppose that **at equilibrium** the concentration $c_{N_2,\,eq}$ (in mol/L) of dissolved N_2 within a living tissue is proportional to the **partial** pressure P_{N_2} in N_2 surrounding this tissue according to Henry's law : $c_{N_2,\,eq} = H \times P_{N_2}$ with $H = 6 \times 10^{-4}$ mol.L^{-1}.bar^{-1}.

II.1 First estimation of the danger

Q8. Recall the approximate value of the molar fraction of N_2 in air, then deduce the approximate value of $c_{N_2,\,eq}(z=0)$ for atmospheric pressure $P_0 = P(z=0)$.

Q9. Recall without any demonstration the hydrostatic pressure profile $P(z)$ within water. Determine the approximate value of $c_{N_2,\,eq}(z_0)$ with $z_0 = 30$ m.

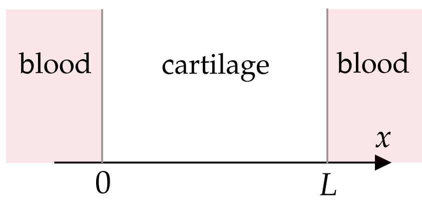
We imagine that the diver, initially at equilibrium at $z_0 = 30$ m of depth, suddenly emerges at $z = 0$. The total volume of blood of a human being is about $V = 5$ L. For the following, to get a rough estimation we consider only the blood, as a closed system. The ideal gas constant will be noted R_{ig} with $R_{ig} = 8.3$ J.K^{-1}.mol^{-1}.

Q10. Determine the amount (in moles) of N_2 gas that appears within the diver's blood if it instantaneously reaches its new equilibrium. Using ideal gas law, convert this amount of N_2 gas into a volume of gas at atmospheric pressure, is this volume enough to obstruct a blood vessel ?

II.2 Avoid the accident : the slow diffusion of dinitrogen in living tissues

Usually gas bubbles do not emerge in the blood, which circulates very often through the lungs and thus adapts quickly its N_2 concentration with the surrounding pressure $P(z)$. Therefore we suppose that in blood for each depth z the concentration in N_2 is the one at equilibrium stated by Henry's law. The diver just reached the surface consequently that concentration is : $c_{N_2,\,eq} = 5 \times 10^{-4}$ mol/L.

However, in tissues such as cartilage, diffusion limits the transport of N_2 : it takes time for it to diffuse and reach new equilibrium. Note that N_2 is **not** produced nor consumed by cartilage or any living tissue. We call $n(x,t)$ the number of gas N_2 molecules per m^3, and name D the diffusion coefficient of N_2 in cartilage.



Q11. By continuity of $n(x,t)$ in 0 and L, determine the numerical values of $n(0)$ and $n(L)$.

Q12. Without any demonstration, express the differential equation that $n(x,t)$ follows here for $x \in [0, L]$.

We look for stationary solutions for $n(x,t) - n(0)$. The plane at $x = \frac{L}{2}$ is Π^+ (a symmetry plane) for both $n(x,t)$ and $\vec{j}_{N_2}$.

Q13. Demonstrate rigorously that $n(x,t)$ can be written as :

$$n(x,t) = n(0) + Ae^{-q_m^2 Dt}\sin(q_m x) \qquad \text{with } q_m = \frac{\pi}{L}(1+2m) \text{ for } m \in \mathbb{N} \text{ and } A \text{ an unknown.}$$

Q14. Determine the only value for m that makes sense physically, and justify why by representing graphically $n(x,t)$ for different m.

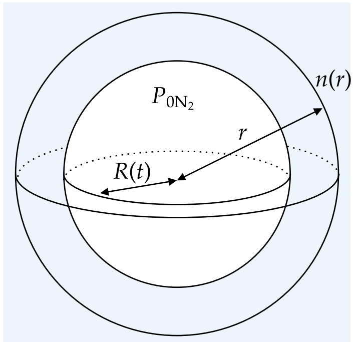
At $t = 0$, the maximum of concentration of N_2 within the cartilage is $c_{N_2, eq, z=z_0} = 2 \times 10^{-3}$ mol/L. The diffusion coefficient of N_2 in cartilage is $D = 2 \times 10^{-9}$ m^2.s^{-1} and $L = 5$ mm.

Q15. Determine the numerical value of A, then define a characteristic time τ for the decay of this excess of concentration in N_2 within cartilage : how long should a diver typically take to reach the surface after diving at 30 meters ?

II.3 Growth of dinitrogen bubbles in living tissues

We study an isolated bubble of N_2 of partial pressure P_{0N_2} and radius $R(t)$. The evolution of the radius $R(t)$ of the N_2 bubble is slow enough for the diffusion of N_2 in the liquid to be in steady-state. n is the number of dissolved molecules of N_2 per m^3 within the living tissue ($\sim$ water) with $n(r) \underset{r\to\infty}{\longrightarrow} n_\infty$. D is the diffusion coefficient of the dissolved N_2 within the liquid and V_n the molar volume of the gas, **supposed to be constant**. We neglect any presence of dioxygen.
To study $R(t)$ we neglect surface tension and give the following law :

- Henry's law : $n(R) = HP_{0N_2}$ with $H = 3.6 \times 10^{17}$ kg^{-1}.s^2.m^{-2}



In spherical coordinates for a scalar field $n(r)$ the gradient and Laplacian are written as such :

$$\overrightarrow{\text{grad}}\, n = \frac{\partial n}{\partial r}\overrightarrow{e_r} \qquad \Delta n = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial n}{\partial r})$$

Q16. Determine $n(r)$ using r, R, $n(R)$ and n_∞.

Q17. Using Fick's law, determine the volume variation rate $\dot{V}$ of the bubble per unit of time.

Q18. Show that the bubble's radius $R(t)$ follows :

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{DV_n}{\mathcal{N}_A R(t)}\left(n_\infty - HP_{0N_2}\right)$$

Here $n_\infty = 1.1 \times 10^{23}$ m^{-3}, $T = 310$ K, $D = 2 \times 10^{-9}$ m^2.s^{-1}, $R_{ig} = 8.3$ J.K^{-1}.mol^{-1} and $P_{0N_2} = 0.8$ bar.

Q19. Demonstrate that the bubble will indeed grow, then define and determine the value of the duration Δt for it to grow to $R_0 = 1$ cm. Comment your result in regards of previous results for diffusion of dissolved N_2 in cartilage.

In our model, the diver suddenly reached the surface : the concentration of dissolved N_2 initially remained the one at equilibrium for $z = z_0$ (before the diffusion of the II.2 significantly occurs), but the partial pressure P_{0N_2} in N_2 dropped causing this rapid bubble growth. To avoid such a dangerous drop, divers ascend from the depths in discrete stages, stopping during tens of minutes (this duration depending notably on the maximum depth and on the duration spent at this maximum depth).